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AN EFFICIENCY AND EFFICACY ANALYSIS OF SOME APPROACHES FOR SOLVING A FUZZY LINEAR MULTI-CRITERIA PROBLEM

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Resumé

L'analyse de l'efficience et de l'efficacité constitue une notion clé pour l'étude d'un algorithme. Dans le contexte des problèmes linéaires multicritères flous, plusieurs algorithmes ou approches existent dans la littérature. Cette étude a pour objectif d'analyser la complexité et l'efficacité de certaines approches existantes. Nous rappelons plusieurs approches, des plus récentes aux plus anciennes. Nous analysons la complexité de chacune d'entre elles. Enfin, l'efficacité est évaluée à travers un exemple numérique. Nos résultats révèlent que l'approche basée sur la fonction d'erreur est plus efficace que les autres, mais pas nécessairement plus efficiente que les approches existantes rappelées.

Mots clés : Analyse de l'efficience, Solution de compromis, Analyse de l'efficacité, Fonction d'erreur, Fonction d'aide à la décision

Abstract

The efficiency and efficacy analysis are the key notions for analyzing an algorithm. In the context of fuzzy linear multi-criteria problem, several algorithms or approaches exist in literature. The study aimed to analyze the complexity and effectiveness of some existing approaches. We recall some existing approaches, from the recent one to the old one. We analyze the complexity of each of them. Finally the efficacy has been analyzed through a numerical example. Our findings reveals that, the approach based on errors function is more efficacy than others but not necessary efficient than these existing recalled approaches.

Key words : Efficiency analysis, Compromise Solution, Efficacy analysis, Error Function, Decision-Making Function

1. Introduction

The methods of solving a linear multi-criteria programming requires that the data determining a linear problem be real numbers without vagueness. However, in some situations, the data describes a real is often subject of vagueness [1] [2]. This justifies the inefficient information or incomplete information that an investor can receive from the real world.

To reflect this vagueness due to the lack of data, the fuzzy set theory introduced by [3], will be used. The fuzzy set theory provides a framework in which vague concepts can be precisely and rigorously studied in order to take into account the imprecision of data [13]. Some authors proposed approaches for solving fuzzy linear multi criteria problems. One can see [4, 5, 6, 7, 8, 9]. The vagueness in a multi criteria problem consists of systematic substitution of fuzzy data for deterministic one into the multi criteria problem. In 2018, Y.T. Mangongo et al., [1] used fuzzy concepts to propose a fuzzy multi criteria model to truck the portfolio selection problem. Y.T. Mangongo and J.D.B. Kampempe [5] proposed two approaches to bring a balance between effectiveness and efficiency while solving a multiobjective programming problem with fuzzy objective functions. In the first approach, they used the Nearest Interval Approximation Operator for fuzzy numbers to propose the deterministic counterpart of the original fuzzy problem. In their second approach, they used the Embedding Theorem for fuzzy numbers. More recently, J.L. Makubikua et al. [13] proposed a new approach based on errors function.

The aim of this paper is to study the complexity and efficacy of some existing approaches. For this purpose, we recall some existing approaches, from the recent up to the old one. A numerical example is done in order to analyze the effectiveness (efficacy) of these approaches

The paper is organized as follows: the second section will be devoted to preliminaries concepts. In the third section, we present some existing approaches. Then in section 4, we perform the efficiency and efficacy analysis. Finally, the paper will be ended by some concluding remarks.

2. Preliminaries

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Definition 1 [3] *Let X be a non-empty set. A fuzzy set, denoted by $\tilde{A}$ is defined as follow:*

$$\tilde{A} = \{(x, f_{\tilde{A}}(x)) : x \in X\},$$

where
$f_{\tilde{A}}$ is the membership function of a fuzzy set $\tilde{A}$, defined on X and takes values on the closed interval $[0,1]$.
A fuzzy measure f is a function defined on a power set $\mathcal{P}(X)$ and takes values into $[0, 1]$, satisfying the following conditions [10, 11, 12]:

- Limits cases: $f(\emptyset) = 0$ and $f(X) = 1$
- Monotonicity: for all set $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$ then $f(A) \leq f(B)$
- Continuity: for any nested part $A_1 \subset A_2 \subset \cdots \subset A_n$ or $A_n \subset A_{n-1} \subset \cdots \subset A_1$, we have : $lim_{n\to+\infty} f(A_n) = f(lim_{n\to+\infty}(A_n))$

Definition 2 [12] *A fuzzy linear multi criteria problem is formulated as follows:*

$$\begin{cases} \text{"min"}\ (\widetilde{c_1}x, \widetilde{c_2}x, \cdots, \widetilde{c_k}x) \\ \sum_{j=1}^{n} \widetilde{a_{ij}} \odot x_j \leq \widetilde{b_i} \quad ; \\ x_j \geq 0; 1 \leq j \leq n; 1 \leq i \leq m \end{cases} \tag{1}$$

where $\widetilde{c_j}$ is the $(1 \times n)$ dimension matrix; x is the $(n \times 1)$ dimension matrix and $\widetilde{c_{rj}}$, $\widetilde{a_{ij}}$, $\widetilde{b_j}$ the fuzzy intervals with the respective membership functions $\mu_{\widetilde{c_{rj}}}$, $\mu_{\widetilde{a_{ij}}}$, $\mu_{\widetilde{b_j}}$ with $1 \leq r \leq k$. Additionally $\odot$ expresses the multiplication of fuzzy numbers.

3. Some existing approaches

3.1. Approach based on errors function [13]

Problem (1) is transformed into a deterministic one by defining the level set α. We obtain therefore the following deterministic problem:

$$\begin{cases} \text{"max"}\ (c_1x, c_2x, \cdots, c_kx) \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i \quad ; \\ x_j \geq 0; 1 \leq j \leq n; 1 \leq i \leq m \end{cases} \tag{2}$$

After that, we solve separately each problem with one criteria and we compute f_r^a ; which is the value of a criteria r to a level α . If one problem does not have a solution; we should revise the value of α . After that, we compute for each criteria the error function defined as follow:

$$u_r(x) = \frac{f_r^a - f_r(x) + E_r}{f_r^a + \varepsilon}; \tag{3}$$

where ε is an arbitrary positive constant, $f_r(x) = c_r x$ (criteria), E_r the deviation variable of a given criteria with the denominator $f_r^a + \varepsilon \neq 0$ for $1 \leq r \leq k$.

In order to find the compromise solution of Problem (1), we minimize the sum of all these errors functions under the following constraints:

$$\begin{cases} 0 \leq u_r(x) \leq 1 \\ f_r^a - E_r \leq f_r(x) \leq f_r^a + E_r \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i \\ x_j \geq 0; 1 \leq j \leq n; 1 \leq i \leq m \end{cases} . \tag{4}$$

For more details, readers can be referenced on the cited source.

3.2. Sakawa-Yano's approach [8]

This approach is very interesting because it involves fuzzy logic at all levels of the problem (on the parameters of criteria and on the parameters of constraints).
The method starts by a fuzzy problem which will be transformed by a deterministic one by assigning an element of a level set α of the form:

$$\begin{cases} \text{"min"}(f_1(x), \cdots, f_k(x)) \\ g_i(x) \leq 0 \quad ; \\ x_j \geq 0 \end{cases} \tag{5}$$

where

$$f_i(x) = c_i(x)$$

and

$$g_i(x) = \sum_j a_{ij}x_j - b_i.$$

Under constraints $g_i(x) \leq 0$, we maximize and minimize, each criteria separately to obtain an interval of variation of the value of criteria i. One defines the membership functions as follows:

$$\mu_i(x) = \frac{f_{i1} - f_i(x)}{f_{i1} - f_{i0}}; \tag{6}$$

where f_{i0} is the less interesting value of the criteria and f_{i1} is the most interesting value of the criteria i. One defines the decision-making functions DM_i as follows:

$$DM_i(x) = \begin{cases} 0, & if\ \mu_i(x) \leq 0 \\ \mu_i(x), & if\ 0 < \mu_i(x) < 1 \\ 1, & if\ \mu_i(x) \geq 1 \end{cases} \tag{7}$$

where $DM_i(x) = 1$ when the objective is reached and $DM_i(x) = 0$ when the objective is not reached. Finally, we maximizes the DM_i functions. If there is no solution, or if the solution does not satisfy the investor, then we must change the membership functions, and return to the previous step. For further information, readers can be referenced on the cited paper.

3.3. Reardon's approach [6]

This approach transforms the initial Problem (1) into the following deterministic one:

$$\begin{cases} \min \begin{cases} f_1(x) \\ \cdots \\ f_k(x) \end{cases} \\ g_1(x), \cdots, g_m(x) \leq 0 \\ x \in \mathbb{R}^n \quad ; \\ (f_1(x) \cdots, f_k(x)) \in \mathbb{R}^k \\ (g_1(x), \cdots, g_m(x)) \in \mathbb{R}^m \\ x \geq 0 \end{cases} \tag{8}$$

For each criterion, one should define a membership function called the Reardon's membership function. Reardon define the expressions of his membership functions as follows:

- If $f_i \leq (O_i - E_i)$ then:

$$f'(f_i) = \left[\frac{S_{max}}{f_{i\,min} - (O_i - E_i)}\right] \cdot (f_i - (O_i - E_i)). \tag{9}$$

- If $(O_i - E_i) \leq f_i \leq (O_i + E_i)$ then:
  $f'(f_i) = 0.$ (10)

- If $(O_i + E_i) \leq f_i$ then:

$$f'(f_i) = \left[\frac{S_{min}}{(O_i + E_i) - f_{i\,max}}\right] \cdot \left(f_i - (O_i + E_i)\right). \quad (11)$$

Comparing with the classical methods of fuzzy optimisation, in this algorithm the level is in inverse. Instead of having 1 for the value of membership function when f_i is belongs to the set of desired values (that means in the interval $[O_i - E_i, O_i + E_i]$), the value of membership function is null. This inversion of the degree of belonging allows us to obtain very simple global criteria which aggregate all criteria

$$F = \frac{1}{k}\sum_{i=1}^{k} f'(f_i) \quad (12)$$

Function F is the mean of the Reardon's membership functions of criteria. All these Reardon's membership functions has been normalized by an intermediate of S_{max} and of S_{min}. Interested readers can see [6] for more details.

3.4. Approach based on statistics methods [14]

In this approach, authors build a function called "aggregation function". A function that aggregates all criteria after the defuzzification process. They consider a fuzzy linear multi-criteria problem where parameters are triangular fuzzy numbers, defined as follow:

$$\begin{cases} \text{"max"}\ 2^i = \sum_{j=1}^{n} \tilde{c}_i x_j \\ \sum_{j=1}^{n} (m_{ij}, l_{ij}, r_{ij}) \odot x_j \le (d_i, e_i, f_i) \\ x_j \ge 0; 1 \le j \le n; 1 \le i \le m \end{cases} \quad (13)$$

The approach is summarized into the four following steps:

1. Deffuzification of Problem (13).
2. Solve each objective function z_i under all constraints by one of the known resolution method (ie. Simplex method).
3. Build the aggregation function as follows:

$$z = \frac{1}{\min \phi_i}\sum_{i=1}^{n} z_i \quad (14)$$

4. Maximize function z under all constraints.

More details can be found here [14].

3.5. Ahmed M.K. approach [15]

This approach considers Problem (1) in which authors assume that constraints are deterministic. This gives:

$$\begin{cases} \text{"min"}\ (\tilde{f}_1(x), \cdots, \tilde{f}_k(x)) \\ s/c \\ x \in X = \{x \in \mathbb{R}^n | g_i(x) \le 0; i = 1, \cdots, m\} \end{cases},$$

where $\tilde{f}_j (j = 1, \cdots, k)$ are fuzzy functions defined on $\mathbb{R}^n$ and $g_i\ (i = 1, \cdots, k)$ are real functions defined on $\mathbb{R}^n$. The Problem (15), can be formulated as follows:

$$\begin{cases} \min \max \rho \sum_{j=1}^{n} \bar{A}_j^l x_j + \left(\sum_{i=1}^{n} \frac{\bar{f}_i(x)}{f_{-i} - \bar{f}_i}\right) \\ s/c \\ x \in X = \{x \in \mathbb{R}^n | g_i(x) \le 0; i = 1, \cdots, q\} \\ \lambda \le 0, x_j \ge 0; i = 1, \cdots, m, j = 1,2, \cdots, k \end{cases},$$

where $f_{-i} = \min \bar{f}_i(x)$, $f_i = \max \bar{f}_i(x)$ and ρ a smallest real number. The Problem (16) can be again formulated as follows:

$$\begin{cases} \min \lambda \\ s/c \\ \sum_{j=1}^{n} \bar{A}_j^l x_j + \left(\sum_{i=1}^{n} \frac{\bar{f}_i(x)}{f_{-i} - \bar{f}_i}\right) f_{-i}\lambda, i = 1,2, \cdots, q\ , \\ x \in X = \{x \in \mathbb{R}^n | g_i(x) \le 0; i = 1, \cdots, q\} \\ \lambda \le 0, x_j \ge 0; i = 1, \cdots, m, j = 1,2, \cdots, k \end{cases}$$

where $\lambda = \max \rho \sum_{j=1}^{n} \bar{A}_j^l x_j + \left(\sum_{i=1}^{n} \frac{\bar{f}_i(x)}{f_{-i} - \bar{f}_i}\right), i = 1, \cdots, m.$

For more details, one can see [15].

3.6. Luhandjula M.K. approach [16]

In this approach, authors used Problem (1) that constraints are deterministic. A compromise solution is found by solving its equivalent deterministic given as follows:

$$\begin{cases} \text{"min"}\ (\bar{f}_1(x), \cdots, \bar{f}_k(x)) \\ s/c \\ x \in X = \{x \in \mathbb{R}^n | g_i(x) \le 0; i = 1, \cdots, m\} \end{cases}$$

Authors used the nearest interval of approximation of a fuzzy number to find an approximation of Problem (18). The equivalent deterministic is therefore:

$$\begin{cases} \text{"min"}\ (F_1(x), \cdots, F_k(x)) \\ s/c \\ x \in X = \{x \in \mathbb{R}^n | g_i(x) \le 0; i = 1, \cdots, m\} \end{cases}$$

where $F_i(x) = [f_i^L(x), f_i^U(x)]$ is the nearest interval approximation of the corresponding fuzzy function $\bar{f}_i$. More details can be found in [16]. In the next section, we perform the efficiency and efficacy analysis of approaches explained in section 3.

4. Efficiency and efficacy analysis

4.1. Efficiency analysis

A/ Approach based on errors function

We have k fuzzy objective functions and m fuzzy constraints. From there, we found the deterministic counterpart on the initial fuzzy problem, which leads us to a complexity of order $O(k + m)$. After that, we solve each objective function subject to all constraints, which gives a complexity of order $O(k)$. We build after the errors functions, each criteria is transformed into an error function, which gives a complexity of order $O(k)$. Finally, we minimize the sum of these errors functions. Putting everything together, we obtain a complexity of order $O(3k + m)$.

(15)

The complexity of this approach depends on the number of criteria that the problem contains. By the definition of error functions, the values of criteria are adjusted by taking into account of criteria with their associated constraints simultaneously.

B/ Luhandjula approach

(16)

For each fuzzy coefficient (in objectives or constraints), the algorithm computes a nearest interval of approximation. If q is the number of fuzzy coefficients (in objectives or constraints), we have a complexity of order $\mathcal{O}(q)$. Once each fuzzy number is replaced by its nearest interval of approximation, the

algorithm generates a system of equations. Let p the number of objective functions and m the number of constraints, we have then
$\mathcal{O}(m + p)$. This gives globally a complexity of order $\mathcal{O}(m + p + q)$. After transformation, we applied simplex algorithm to solve the resulting problem. The simplex algorithm has an exponential complexity in worst case.

C/ Approach based on statistics methods

Let p and n the number of objective functions and constraints respectively. The computation of the means of each objective function give a complexity of order $\mathcal{O}(p)$. The transformation of each fuzzy constraint into a deterministic one gives a complexity of order $\mathcal{O}(m)$. This gives a global complexity of $\mathcal{O}(p + m)$ after transformation. After that, we applied the simplex algorithm which has an exponential complexity in worst case.

D/ Ahmed approach

Let q fuzzy numbers (from objective functions and constraints); each fuzzy number is transformed by a Ranking function; which gives a linear cost proportional to q; therefore a complexity of order $\mathcal{O}(q)$.
In this step, the initial problem is transformed into a deterministic one by a system of equations or inequalities. Let m the number of constraints of the initial problem and p the number of objective functions, we have then a complexity of order $\mathcal{O}(m + p)$. Finally, we have a complexity of order $\mathcal{O}(q + m + p)$. After that, we applied the simplex algorithm which has an exponential complexity in worst case.

E/ Sakawa-Yano approach

Let n the number of variables, m the number of constraints, p the number of objective functions and k the number of iterations. We maximize and then minimize each objective function of the problem; which gives a complexity of order $\mathcal{O}(p + p)$, or we can write $\mathcal{O}(2p)$. After that, we build the approximation functions with complexity $\mathcal{O}(p)$. The constraints are therefore rewritten with a complexity of order $\mathcal{O}(m + n)$. And because the construction of approximation functions and the rewritten of constraints have been made k times, we have then a complexity of order : $\mathcal{O}(2p) + k(\mathcal{O}(p + m + n))$.

F/ Reardon approach

Initially we have k criteria et m fuzzy constraints. We transform them into a deterministic criteria and constraints, which gives a complexity of order $\mathcal{O}(k + m)$. After that, for each deterministic criteria, we construct a membership function, called « Reardon's membership function », which gives a complexity of order $\mathcal{O}(k)$. Finally, we found function F that is the mean of Reardon's membership function, which gives k sum.
Globally, we have a complexity of order $\mathcal{O}(3k + m)$. We applied, after, the simplex algorithm which has an exponential complexity in worst case.

4.2. Efficacy analysis

In order to see the effectiveness of approaches, we perform in this subsection an efficacy analysis.

4.2.1. Problem's Presentation [13]

We consider the following problem which contains two decisions variables $x = (x_1, x_2)$; three criteria $\widetilde{f_1}(x) = \widetilde{c_{11}}x_1 + \widetilde{c_{12}}x_2$, $\widetilde{f_2}(x) = \widetilde{c_{21}}x_1 + \widetilde{c_{22}}x_2$ and $\widetilde{f_3}(x) = 6x_1 + \widetilde{c_{23}}x_2$; two constraints $\widetilde{a_{11}}x_1 + \widetilde{a_{12}}x_2 \leq \widetilde{b_1}$ and $\widetilde{a_{21}}x_1 + \widetilde{a_{22}}x_2 \leq \widetilde{b_2}$. The membership functions of each fuzzy set are defined as below:

$$\mu_{\widetilde{c_{11}}} = \begin{cases} 0, & x \leq 5 \\ \dfrac{x^2 - 25}{24}, & 5 < x \leq 7 \\ 1, & 7 < x \leq 9 \\ \dfrac{100 - x^2}{19}, & 9 < x < 10 \\ 0, & 10 \leq x \end{cases} ; \ \mu_{\widetilde{c_{12}}}$$

$$= \begin{cases} 0, & x \leq 2 \\ \dfrac{x - 2}{5}, & 2 < x \leq 7 \\ 1, & 7 < x \leq 12 \\ \dfrac{196 - x^2}{19}, & 12 < x < 14 \\ 0, & 14 \leq x \end{cases} ;$$

$$\mu_{\widetilde{c_{21}}} = \begin{cases} 0, & x \leq 14 \\ \dfrac{x^2 - 196}{188}, & 14 < x \leq 18 \\ 1, & 18 < x \leq 22 \\ \dfrac{576 - x^2}{92}, & 22 < x < 24 \\ 0, & 24 \leq x \end{cases} ; \ \mu_{\widetilde{c_{22}}}$$

$$= \begin{cases} 0, & x \leq 30 \\ \dfrac{x^2 - 900}{325}, & 30 < x \leq 35 \\ 1, & 35 < x \leq 40 \\ \dfrac{2025 - x^2}{19}, & 40 < x < 45 \\ 0, & 45 \leq x \end{cases} ;$$

$$\mu_{\widetilde{a_{11}}} = \begin{cases} 0, & x \leq 0 \\ 2x, & 0 < x \leq 0.5 \\ 1, & 0.5 < x \leq 2 \\ \dfrac{25 - 5x}{15}, & 2 < x < 5 \\ 0, & 5 \leq x \end{cases} ; \ \mu_{\widetilde{a_{12}}}$$

$$= \begin{cases} 0, & x \leq 0 \\ 5x, & 0 < x \leq 0.2 \\ 1, & 0.2 < x \leq 1 \\ \dfrac{9 - 3x}{6}, & 1 < x < 3 \\ 0, & 3 \leq x \end{cases} ;$$

$$\mu_{\widetilde{a_{21}}} = \begin{cases} 0, & x \le 0 \\ 4x, & 0 < x \le 0.25 \\ 1, & 0.25 < x \le 5 \\ \dfrac{64-8x}{24}, & 5 < x < 8 \\ 0, & 8 \le x \end{cases} ; \ \mu_{\widetilde{a_{22}}}$$

$$= \begin{cases} 0, & x \le 5 \\ \dfrac{2x-10}{2}, & 5 < x \le 6 \\ 1, & 6 < x \le 7 \\ \dfrac{64-8x}{8}, & 7 < x < 8 \\ 0, & 8 \le x \end{cases} ;$$

$$\mu_{\widetilde{b_1}} = \begin{cases} 0, & x \le 12 \\ \dfrac{x-12}{3}, & 12 < x \le 15 \\ 1, & 15 < x \le 18 \\ \dfrac{42-2x}{6}, & 18 < x < 21 \\ 0, & 21 \le x \end{cases} ; \ \mu_{\widetilde{b_2}}$$

$$= \begin{cases} 0, & x \le 56 \\ \dfrac{2x-112}{24}, & 56 < x \le 68 \\ 1, & 68 < x \le 74 \\ \dfrac{258-3x}{36}, & 74 < x < 86 \\ 0, & 86 \le x \end{cases} ;$$

$$\mu_{\widetilde{c_{23}}} = \begin{cases} 0, & x \le 5 \\ x-5, & 5 < x \le 6 \\ -x+7, & 6 < x \le 7 \\ 0, & 10 \le x \end{cases}$$

The problem described above can be formulated as follow:

$$\begin{cases} \text{Max } \tilde{f}(x) \equiv \text{"max"} \begin{cases} \widetilde{f_1}(x) = \widetilde{c_{11}}x_1 + \widetilde{c_{12}}x_2 \\ \widetilde{f_2}(x) = \widetilde{c_{21}}x_1 + \widetilde{c_{22}}x_2 \\ \widetilde{f_3}(x) = 6x_1 + \widetilde{c_{23}}x_2 \end{cases} \\ \widetilde{a_{11}}x_1 + \widetilde{a_{12}}x_2 \le \widetilde{b_1} \\ \widetilde{a_{21}}x_1 + \widetilde{a_{22}}x_2 \le \widetilde{b_2} \\ x \ge 0 \end{cases} \quad (20)$$

4.2.2. Resolution by Sakawa's Approach

A/ Deterministic Equivalent Problem

By defining the level set α for $\alpha = 0.8$, we obtain the following deterministic problem:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9.2x_1 + 12.43x_2 \\ f_2(x) = 22.4x_1 + 41.1x_2 \\ f_3(x) = 6x_1 + 6.2\,x_2 \end{cases} \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0 \end{cases} \quad (21)$$

By minimizing each criteria separately and maximizing them on the same way, we obtain than:

$$0 \le f_1(x) \le 131.9$$
$$0 \le f_2(x) \le 436.12$$
$$0 \le f_3(x) \le 68.71$$

B/ Definition of Criteria's Memberships Functions

We should first investigate for the less interesting values and the more interesting values for each criteria. So for this, we need to define the deterministic problems as below respectively for $\alpha = 0$ and $\alpha = 1$:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 10x_1 + 14x_2 \\ f_2(x) = 24x_1 + 45x_2 \\ f_3(x) = 6x_1 + 7\,x_2 \end{cases} \\ 5x_1 + 1.3x_2 \le 21 \\ 8x_1 + 8x_2 \le 86 \\ x \ge 0 \end{cases} \quad (22)$$

and

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9x_1 + 12x_2 \\ f_2(x) = 22x_1 + 40x_2 \\ f_3(x) = 6x_1 + 6x_2 \end{cases} \\ 2x_1 + 1x_2 \le 18. \\ 5x_1 + 7x_2 \le 74 \\ x \ge 0 \end{cases} \quad (23)$$

Hence by solving these problems, we found: $f_{10} = 98$, $f_{20} = 341$ and $f_{30} = 49$ which are the less interesting values of criterion. $f_{11} = 129.33$, $f_{21} = 422.86$ and $f_{31} = 73.33$ which are the more interesting value of criterion.

Now we define the membership functions as follow:

$$\mu_1(x) = \frac{129.33 - 9.2x_1 - 12.43x_2}{129.33 - 98} = 4.13 - 0.2936x_1 - 0.3967x_2.$$

$$\mu_2(x) = \frac{422.86 - 22.4x_1 - 41.1x_2}{422.86 - 314} = 3.8844 - 0.2058x_1 - 0.377x_2.$$

$$\mu_3(x) = \frac{73.33 - 6x_1 - 6.2x_2}{73.33 - 49} = 3.014 - 0.2466x_1 - 0.2548x_2.$$

C/ Definition of Decision Making Functions DM_i

For each criteria, we have these decision making functions:

$$DM_1(x) = \begin{cases} 0, & \text{if } 4.13 - 0.2936x_1 - 0.3967x_2 \le 0 \\ 4.13 - 0.2936x_1 - 0.3967x_2, & \text{if } 0 < 4.13 - 0.2936x_1 - 0.3967x_2 < 1 \\ 1, & \text{if } 4.13 - 0.2936x_1 - 0.3967x_2 \ge 1 \end{cases}$$

$$DM_2(x) = \begin{cases} 0, & \text{if } 3.8844 - 0.2058x_1 - 0.377x_2 \le 0 \\ 3.8844 - 0.2058x_1 - 0.377x_2, & \text{if } 0 < 3.8844 - 0.2058x_1 - 0.377x_2 < 1 \\ 1, & \text{if } 3.8844 - 0.2058x_1 - 0.377x_2 \ge 1 \end{cases}$$

$$DM_3(x) = \begin{cases} 0, & \text{if } 3.014 - 0.2466x_1 - 0.2548x_2 \le 0 \\ 3.014 - 0.2466x_1 - 0.2548x_2, & \text{if } 0 < 3.014 - 0.2466x_1 - 0.2548x_2 < 1 \\ 1, & \text{if } 3.014 - 0.2466x_1 - 0.2548x_2 \ge 1 \end{cases}$$

D/ Maximisation of Functions DM_i

We solve the problem below:

$$\begin{cases} \max\ \mu_D(\mu_1,\mu_2,\mu_3) \\ 0<\mu_1<1 \\ 0<\mu_2<1 \\ 0<\mu_3<1 \\ 26x_1+1.4x_2\le 18.6 \\ 56x_1+7.2x_2\le 76.4 \\ x_1,x_2\ge 0 \end{cases} \quad (24)$$

By the use of aggregation function min (μ_1,μ_2,μ_3), the Problem (18) becomes:

$$\begin{cases} \text{"max" min } (\mu_1,\mu_2,\mu_3) \\ 0<\mu_1<1 \\ 0<\mu_2<1 \\ 0<\mu_3<1 \\ 26x_1+1.4x_2\le 18.6 \\ 56x_1+7.2x_2\le 76.4 \\ x_1,x_2\ge 0 \end{cases} \quad (25)$$

Applying the techniques for solving the max-min problem, the Problem (19) can be transformed into:

$$\begin{cases} \max\lambda \\ \lambda\le\mu_1 \\ \lambda\le\mu_2 \\ \lambda\le\mu_3 \\ 0<\mu_1<1 \\ 0<\mu_2<1 \\ 0<\mu_3<1 \\ 26x_1+1.4x_2\le 18.6 \\ 56x_1+7.2x_2\le 76.4 \\ 0\le\lambda\le 1 \\ x_1,x_2\ge 0 \end{cases} \quad (26)$$

Replacing each criteria by the Sakawa's membership functions, we have the following Problem:

$$\begin{cases} \max\lambda \\ \lambda\le 4.13-0.2936x_1-0.3967x_2 \\ \lambda\le 3.8844-0.2058x_1-0.377x_2 \\ \lambda\le 3.014-0.2466x_1-0.2548x_2 \\ 0<4.13-0.2936x_1-0.3967x_2<1 \\ 0<3.8844-0.2058x_1-0.377x_2<1 \\ 0<3.014-0.2466x_1-0.2548x_2<1 \\ 26x_1+1.4x_2\le 18.6 \\ 56x_1+7.2x_2\le 76.4 \\ 0\le\lambda\le 1 \\ x_1,x_2\ge 0 \end{cases} \quad (27)$$

After solving the Problem (21), we get the following solution

$$\lambda_1=\frac{5765551}{536870}; x_1=\frac{28063}{39281}\approx 0.7144 \text{ and } x_2=\frac{569735}{77396}\approx 7.3613$$

In this case, the compromise solution is $(x_1,x_2)=(0.7144;0.73613)$ with

$$f_1\approx 98.0734; f_2\approx 318.552 \text{ and } f_3\approx 49.9265.$$

In the case where the solution does not meet the investor's interest, we should change the membership functions otherwise the founded solution is the compromise solution of Problem (21).

4.2.3. Resolution by Reardon's Approach

A/ Deterministic Equivalent Problem

For $\alpha = 0.8$; the deterministic equivalent problem of Problem (14) is given below:

$$\begin{cases} \text{Max } f(x)\equiv \text{"max"} \begin{cases} f_1(x)=9.2x_1+12.43x_2 \\ f_2(x)=22.4x_1+41.1x_2 \\ f_3(x)=6x_1+6.2\,x_2 \end{cases} \\ 2.6x_1+1.4x_2\le 18.6 \\ 5.6x_1+7.2x_2\le 76.4 \\ x\ge 0 \end{cases} \quad (28)$$

We set that the fuzzy scale factors max and min are given respectively by: $S_{max}=0.8$ and $S_{min}=0.5$. The minimal and maximal values of criterion are given by:

$$f_{1\,min}=0, \quad f_{1\,max}=131.9;$$
$$f_{2\,min}=0, \quad f_{2\,max}=436.12;$$
$$f_{3\,min}=0, \quad f_{3\,max}=68.71.$$

After that, we define the experimental values of each criteria:

$$O_1=131.9; O_2=436.12 \text{ and } O_3=68.71.$$

We define also the margin errors for each experimental value of criterion:

$$E_1=2; E_2=5; E_3=5.$$

B/ Definition of Reardon's Membership Functions

By defining these, we want that:

$$f_1(x)\le(131.9-2); f_2(x)\le(436.12-5) \text{ and } (68.71-5)\le f_3(x)\le(68.71+5).$$

We have then

$$f_1(x)\le 129.9; f_2(x)\le 431.12 \text{and } 63.71\le f_3(x)\le 73.71.$$

We have now the following Reardon's membership functions:

- $f'(f_1)=\frac{0.8}{-129.9}(9.2x_1+12.43x_2-129.9)=-0.057x_1-0.077x_2+0.8;$
- $f'(f_2)=\frac{0.8}{-431.12}(22.4x_1+41.1x_2-431.12)=-0.0416x_1-0.0781x_2+0.8;$
- $f'(f_3)=0.$

Hence, the mean of these Reardon's membership functions is given by:

$$F=\frac{1}{3}\sum_{i=1}^{3}f'(f_i)=-0.0329x_1-0.05173x_2+0.5333.$$

C/ Minimization of F

The minimization problem of F is given by:

$$\begin{cases} \min\beta \\ \beta+0.0329x_1+0.05173x_2=0.5333 \\ 0.057x_1+0.077x_2\le 0.8 \\ 0.057x_1+0.077x_2\ge -0.2 \\ 0.0416x_1+0.0781x_2\le 0.8 \\ 0.0416x_1+0.0781x_2\ge -0.2 \\ 2.6x_1+1.4x_2\le 18.6 \\ 5.6x_1+7.2x_2\le 76.4 \\ 9.2x_1+12.43x_2\le 129.9 \\ 22.4x_1+41.1x_2\le 431.12 \\ 6x_1+6.2\,x_2\ge 63.71 \\ 6x_1+6.2\,x_2\ge 73.71 \\ x_1,x_2,\beta\ge 0 \end{cases} \quad (29)$$

After solving the above problem, we get the solution below:

$$x_1=\frac{114273}{176630}\approx 0.647; \quad x_2=\frac{9583540}{968247}\approx 9.898$$

with

$$f_1\approx 128.98; f_2\approx 421.3006; \ f_3\approx 65.25.$$

Thus the founded solution is the compromise solution of Problem (14).

4.2.4. Resolution by Approach Based on Errors Function

A/ Deterministic Equivalent Problem

For this approach, we will solve the problem with two different degrees of possibilities, $\alpha = 0.8$ and $\alpha = 1$.

- For $\alpha = 0.8$, we have this below deterministic equivalent problem:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9.2x_1 + 12.43x_2 \\ f_2(x) = 22.4x_1 + 41.1x_2 \\ f_3(x) = 6x_1 + 6.2\,x_2 \end{cases} \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0 \end{cases} \quad (30)$$

- For $\alpha = 1$, we have this below deterministic equivalent problem:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9x_1 + 12x_2 \\ f_2(x) = 22x_1 + 40x_2 \\ f_3(x) = 6x_1 + 6\,x_2 \end{cases} \\ 2x_1 + 1x_2 \le 18 \\ 5x_1 + 7x_2 \le 74 \\ x \ge 0 \end{cases} \quad (31)$$

B/ Determination of f_r^a

- For $\alpha = 0.8$, we solve the problems below:

$$\begin{cases} \max f_r(x) \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0 \end{cases} \qquad r = \{1,2,3\}.$$

After solving these problems, we obtain:

$f_1^{0.8} = \frac{237413}{1800} \approx 131.9$, $f_2^{0.8} = \frac{26167}{60} \approx 436.9$ and $f_3^{0.8} = \frac{46721}{680} \approx 68.71$.

With $\left(0, \frac{191}{18}\right)$ solution of problem for $r = 1$ and $r = 2$, and $\left(\frac{337}{136}, \frac{1181}{136}\right)$ solution of problem for $r = 3$.

- For $\alpha = 1$, we solve the problems below:

$$\begin{cases} \max f_r(x) \\ 2x_1 + 1x_2 \le 18 \\ 5x_1 + 7x_2 \le 74 \\ x \ge 0 \end{cases} \qquad r = \{1,2,3\}.$$

After solving these problems, we obtain:

$f_1^1 = \frac{388}{3} \approx 129.33$, $f_2^{0.81} = \frac{2960}{7} \approx 422.86$ and $f_3^1 = \frac{220}{3} \approx 73.33$.

With $\left(\frac{52}{9}, \frac{58}{9}\right)$ solution of problem for $r = 1$ and $r = 2$, and $\left(0, \frac{74}{7}\right)$ solution of problem for $r = 3$.

C/ Definition of Errors Functions $u_r(x)$ and Minimisation of Their Sum

We do this for two different values of α. In these two cases, we assume that $\varepsilon = 1$.

- For $\alpha = 0.8$, we have:

$$u_1(x) = \frac{131.9 - 9.2x_1 - 12.43x_2 + E_1}{132.9} = 0.9925 - 0.069x_1 - 0.0935x_2 + 0.0075E_1$$

$$u_2(x) = \frac{436.12 - 22.4x_1 - 41.1x_2 + E_2}{437.12} = 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2$$

$$u_3(x) = \frac{68.71 - 6x_1 - 6.23x_2 + E_3}{69.71} = 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_3$$

The sum gives:

$$\sum_{r=1}^{3} u_r(x) = 2.9759 - 0.2065x_1 - 0.2764x_2 + 0.0075E_1 + 0.0023E_2 + 0.0143E_3.$$

Hence, we have this problem which minimize the sum of $u_r(x)$:

$$\begin{cases} \text{"min"}\ \beta \\ \beta + 0.2065x_1 + 0.2764x_2 - 0.0075E_1 - 0.0023E_2 - 0.0143E_3 = 2.9759 \\ 0.9925 - 0.069x_1 - 0.0935x_2 + 0.0075E_1 \geq 0 \\ 0.9925 - 0.069x_1 - 0.0935x_2 + 0.0075E_1 \leq 1 \\ 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2 \geq 0 \\ 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2 \leq 1 \\ 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_3 \geq 0 \\ 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_3 \leq 1 \\ 9.2x_1 + 12.43x_2 + E_1 \geq 131.9 \\ 9.2x_1 + 12.43x_2 - E_1 \leq 131.9 \\ 22.4x_1 + 41.1x_2 + E_2 \geq 431.12 \\ 22.4x_1 + 41.1x_2 - E_2 \leq 431.12 \\ 6x_1 + 6.2\,x_2 + E_3 \geq 68.71 \\ 6x_1 + 6.2\,x_2 - E_3 \leq 68.71 \\ 2.6x_1 + 1.4x_2 \leq 18.6 \\ 5.6x_1 + 7.2x_2 \leq 76.4 \\ x_1, x_2 \geq 0 \end{cases} \tag{32}$$

After solving this problem, we obtain the following solution:

$$(\beta, x_1, x_2, E_1, E_2, E_3) = \left(\frac{455239}{5368700}; 0; \frac{71020}{6693}; \frac{67133}{13985800}; \frac{1}{391}; \frac{2629}{900}\right).$$

Thus the compromise solution of the Problem (14) is given by: $(x_1, x_2) = \left(0; \frac{71020}{6693}\right)$ with

$$f_1 \approx 131.896;\ f_2 \approx 436.11;\ f_3 \approx 65.79.$$

- For $\alpha = 1$, we have:

$$u_1(x) = \frac{129.33 - 9x_1 - 12x_2 + E_1}{130.33} = 0.9923 - 0.069x_1 - 0.0921x_2 + 0.0077E_1$$

$$u_2(x) = \frac{422.86 - 22x_1 - 40x_2 + E_2}{423.86} = 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2$$

$$u_3(x) = \frac{73.33 - 6x_1 - 6x_2 + E_3}{74.33} = 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_3$$

The sum gives:

$$\sum_{r=1}^{3} u_r(x) = 2.9765 - 0.2017x_1 - 0.2672x_2 + 0.0077E_1 + 0.0024E_2 + 0.0135E_3.$$

Hence, we have this problem which minimize the sum of $u_r(x)$:

$$\begin{cases} \text{"min"}\ \beta \\ \beta + 0.2017x_1 + 0.2672x_2 - 0.0077E_1 - 0.0024E_2 - 0.0135E_3 = 2.965 \\ 0.9923 - 0.069x_1 - 0.0921x_2 + 0.0077E_1 \geq 0 \\ 0.9923 - 0.069x_1 - 0.0921x_2 + 0.0077E_1 \leq 1 \\ 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2 \geq 0 \\ 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2 \leq 1 \\ 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_3 \geq 0 \\ 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_3 \leq 1 \\ 9x_1 + 12x_2 + E_1 \geq 129.33 \\ 9x_1 + 12x_2 - E_1 \leq 129.33 \\ 22x_1 + 40x_2 + E_2 \geq 422.86 \\ 22x_1 + 40x_2 - E_2 \leq 422.86 \\ 6x_1 + 6\,x_2 + E_3 \geq 73.33 \\ 6x_1 + 6\,x_2 - E_3 \leq 73.33 \\ 2x_1 + 1x_2 \leq 18 \\ 5x_1 + 7x_2 \leq 74 \\ x_1, x_2 \geq 0 \end{cases} \tag{33}$$

After solving this problem, we obtain the following solution:

$$(\beta, x_1, x_2, E_1, E_2, E_3) = \left(\frac{194607}{10173740}; \frac{2115190}{370977}; \frac{1871510}{287977}; \frac{60881}{2079710}; \frac{8495170}{228277}; \frac{368073}{2899100}\right).$$

Hence the compromise solution of the Problem (14) is given by: $(x_1, x_2) = \left(\frac{2115190}{370977}; \frac{1871510}{287977}\right)$ with

$$f_1 \approx 129.318;\ f_2 \approx 385.44;\ f_3 \approx 73.212.$$

By this numerical example, we see clearly that the Reardon's approach is easily applied and provides an optimal solution by comparing with Sakawa's approach. The Reardon's approach is less complex than Sakawa approach of the limited number of steps for reaching the solution. The common point is that all solutions provide by these two approaches depend on the level α chosen by the investor.

We can see clearly also that the solution obtained by the approach based on errors function is more effective (improved) than the solution obtained by Reardon approach without invoking the one obtained by Sakawa approach. As Sakawa and Reardon approaches, the solution of the approach base on errors function depends also on the choice of a level α by the investor.

5. Concluding Remarks

In this article, we performed an efficiency and efficacy analysis of some existing approaches. The aim was to determine which of them is more efficient and efficacy in terms of faithfulness of the obtained solutions and complexity. The analysis reveals that the solution obtained by the approach based on errors function improves the one obtained by the Sakawa and Reardon approaches.

As Sakawa and Reardon approaches, the approach based on errors function takes into account of fuzzy data in the formulation of the problem inside of constraints and criteria.

The Sakawa approach gives a no satisfactory solution (less effective) compared with solutions from others approaches studied in this paper. The approach based on errors function gives the best solution (more effective) than the solutions from others. The approach based on errors function becomes more complex when the number of criteria increases.

Despite its contributions, this study presents some limitations. First, the efficacy comparison was carried out using a single numerical example with a limited number of criteria and constraints, which may not fully reflect the behavior of the approaches in large-scale problems. Second, as in most fuzzy optimization methods, the compromise solutions depend on the choice of the possibility level α, which introduces a degree of subjectivity into the decision-making process, that can affect the final result.

Future research could focus on testing these approaches on high – dimensional fuzzy multi-criteria problems in order to evaluate scalability and computational performance. Conducting systematic benchmarking experiments would strengthen the validation of theoretical results.

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